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Numerical Simulation of a System of Colliding Bodies in a Gravitational Field

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Three-dimensional gravitating systems of inelastically colliding masses are found in many astrophysical contexts. In this paper, we present a general numerical solution for this type of problem. Simulations with a small number of particles (typically 100) can be made to represent actual systems containing many more particles simply by means of suitable scaling. The main practical problem is to establish whether two given particles collide or not: we have developed a new method, sufficiently general as to apply to any distribution of gravitational field.

I. INTRODUCTION

The study of a three-dimensional gravitating system of colliding particles has many potential astrophysical applications: for example, the dynamics of Saturn's Ring, the formation of the solar system, the flattening of protogalaxies, etc. [1–5]. It was qualitatively shown many years ago that a rotating nebula consisting of a number of bodies undergoing inelastic collisions flattens as a whole and forms a central condensation [6]. Nevertheless, a complete quantitative study has not been carried out. Whilst numerical experiments for colliding systems have stimulated significant progress in molecular dynamics (see, for example [7–11]), corresponding calculations have so far only been carried out in astrophysics by Ulam [12], who was interested in the nuclei of galaxies, and by Trulsen [13, 14], who studied the dynamics of "jet streams". Note in this context that the *mass distribution* of an ensemble of colliding bodies has been studied numerically either by solving analytic equations, or by Monte Carlo methods (see for example [15–18]).

Numerical simulations can be used to study systematically the dynamical evolution of gravitating systems of colliding particles. We present here a general numerical method by means of which this kind of system may be studied. This method has been used to compute a number of models described elsewhere ([1–5] and forthcoming publications). Here we shall discuss this method essentially by means of a simple illustrative model, which we call the "standard model", and which has the

following characteristics: attraction between particles of finite dimensions has been neglected, and so particle orbits are Keplerian around a central mass point. Each collision is assumed to be instantaneous, consequently simultaneous collisions of several particles can be entirely neglected. All particles are spheres having the same mass and radius. The total mass of the particles has been neglected with respect to the central mass. After a collision, the perpendicular component of the relative velocity of two colliding particles is multiplied by a coefficient k which lies between 0 and -1, whilst the grazing component is conserved for this first model.

The method, however, can be extended easily to more complicated situations: for example, a different collision model or different potential fields or even self-gravitating systems. The principal difficulty is to establish whether two given particles will in fact collide or not, and our method is different from Trulsen's [13].

In Section II, we describe the scaling procedure, and in Section III, the choice of initial conditions. A method for solving Kepler's equation is given in Section IV, and in Section V we describe the strategy used to find colliding pairs. The effect of a collision is treated in Section VI. Some practical considerations are given in Section VII, and possible future improvements appear in Section VIII.

II. Number and Size of Particles

The computing time increases at least as the number of potential collision pairs, i.e., it is proportional to N^2, N being the number of particles. However, the quality of the results is hardly likely to improve faster than $N^{1/2}$ (see below). It seems difficult in practice to follow the movement of more than, say, a thousand colliding particles, even using the largest computers currently available. Fortunately, the results for a few hundred particles can be scaled in such a way as to simulate the evolution of a much more realistic system.

In so far as the system may be described in terms of a simple Boltzmann equation, it has been shown [19] that, in the case of inelastic collisions, the classical Boltzmann equation [20] may be written in the form:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\mathbf{x}}{x^3} \cdot \frac{\partial f}{\partial \mathbf{v}} = \int_{\mathbf{v}'} \int_{\substack{\mathbf{u} \\ (\mathbf{v}'-\mathbf{v})\cdot\mathbf{u}>0}} (2r)^2 (\mathbf{v}' - \mathbf{v}) \cdot \mathbf{u} \left(\frac{f' f_1'}{k^2} - f f_1 \right) d\mathbf{v}' \, d\mathbf{u}, \quad (1)$$

where f is the distribution function, r the radius of the particles, t the time, k the rebound coefficient (see Section VI), $\mathbf{x}$ the position, $\mathbf{v}$ the velocity, and $\mathbf{u}$ the unit vector along the direction of the line of centres of two particles at collision. Primed, unprimed, and indexed quantities have the meanings usually found for the Boltzmann equation. Note that this equation is only valid (see [5]) if one neglect the

variation of f over a distance of the order of the dimension of the particles. Putting $fR^2/N = F$, where R is some characteristic dimension of the system, we have:

$$\frac{Df}{Dt} = N\left(\frac{r}{R}\right)^2 \int_{\mathbf{v}'} \int_{\mathbf{u}} 4(\mathbf{v}' - \mathbf{v}) \cdot \mathbf{u} \left(\frac{F'F_1'}{k^2} - FF_1\right) d\mathbf{v}' \, d\mathbf{u} = N\left(\frac{r}{R}\right)^2 \phi(F, F), \qquad (2)$$

$$(\mathbf{v}'-\mathbf{v})\cdot\mathbf{u}>0$$

where $\phi\,(F, F)$ is a second-order collision operator. We transform the time variable by:

$$d\tau = N\left(\frac{r}{R}\right)^2 dt, \qquad (3)$$

and obtain:

$$\frac{DF}{D\tau} = \phi(F, F) \qquad (4)$$

where neither N nor r are explicitly present. Thus, a change of N or r affects only the speed of evolution, which is proportional to $N(r/R)^2$. We thus see that a system consisting of many small particles can be realistically simulated by a system containing fewer, but larger particles; indeed, if $N(r/R)^2$ is the same for both systems, the time scale of evolution is the same. This effect of scaling can be seen from Fig. 1, which has been derived from numerical experiments.

Thus, a given physical system in principle can be simulated by a model with arbitrary values of N and r. In practice, however, there are some limitations:

(1) The procedure does become unsatisfactory if r is too large, because the gravitational force acting on each of two particles which have just collided will not be the same: this effect introduces a distortion which increases towards the centre of the system. This was pointed out by Trulsen [13], who noted that it would tend to increase the spread in eccentricities and semi-major axes of particle orbits.

(2) Too small a number of particles is also unsatisfactory, because statistical fluctuations, proportional to $1/N^{1/2}$, are then too large. Simulations with different values of N are shown in Fig. 2. The initial positions and velocities of particles are generated using a pseudorandom number generator (see Section IV). Keeping all other parameters constant, we started the random number generator at four different values for each value of N. The relative scatter of the curves gives an estimate of the statistical fluctuations.

For 100 particles, the fluctuations are already reasonably small, and we have adopted this value of N in most of our computations. We note also that if more accuracy is desired, it will be more economical to run several 100 particle models and average the results, than to try to increase N, because the computing time increases at least as N^2.

(3) An upper limit to N or r is imposed by the total volume of space initially assigned to the particles: this must clearly be much greater than the total volume of the particles themselves:

$$N(r/R)^3 \ll 1. \tag{5}$$

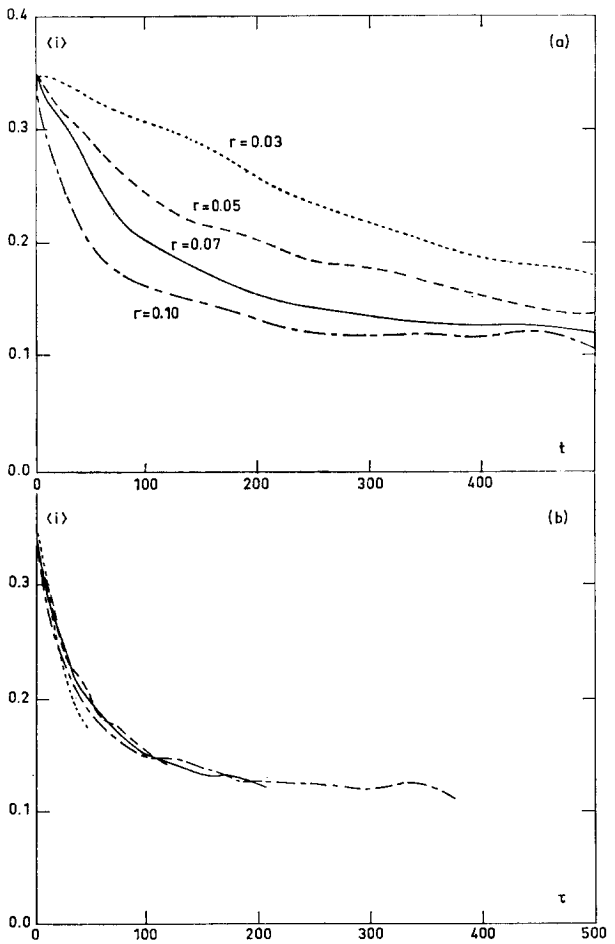

FIG. 1. Variations, as a function of time t (a) and τ (b) respectively, of the mean inclination of the orbital planes for different values of r. The number N of particles is equal to 100 and their radius is $r = 0.03$, 0.05, 0.07 and 0.10 respectively. Initial trajectories are all ellipses, lying between two spheres of radius $R_1 = 1$ and $R_2 = 3$. The initial inclinations of the orbits are distributed between 0 and 0.5 radian. The rebound coefficient k is equal to -0.3. Each curve has been obtained after 5 min. of computing time on the IBM 370-168. We can check that the curves $\langle i \rangle = f(\tau)$ are all similar.

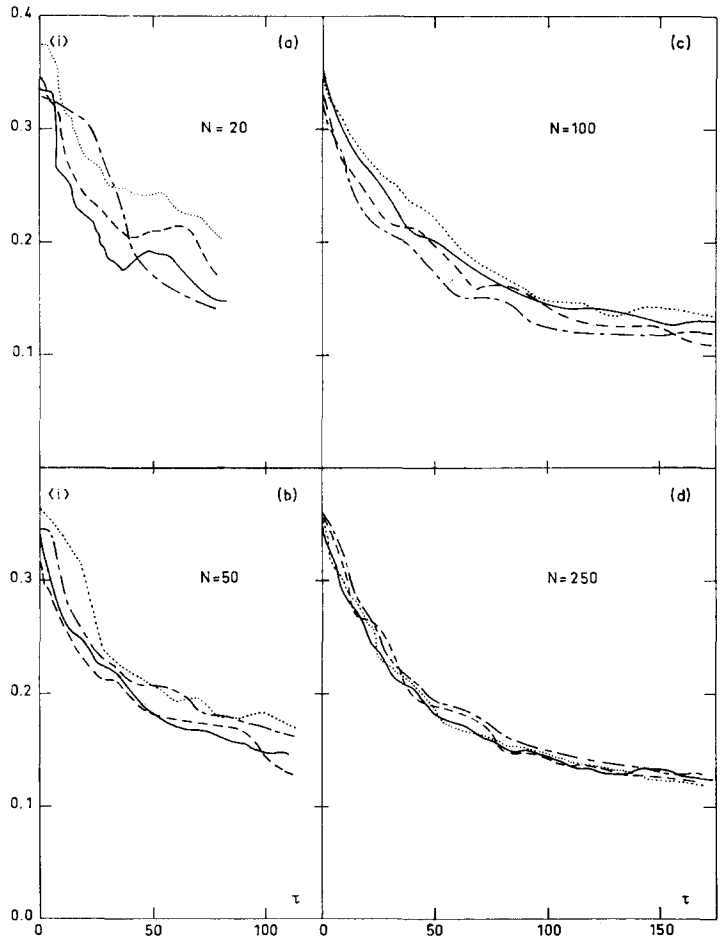

FIG. 2. Variations as a function of time τ of the mean inclination for different values of N and of the initial pseudorandom number generator m. The number N of particles is equal to 20(a), 50(b), 100(c), 250(d) respectively. The starting value of m is respectively equal to 5 (full line), 10 (dashed line), 42 (dotted line) and 0 (dash–dotted line). The radius r of the particles is equal to 0.07. Initial trajectories are all ellipses, which lies between two spheres of radius $R_1 = 1$ and $R_2 = 3$, and centred on the central mass point. The initial inclinations of the orbits are all distributed between 0 and 0.5 radian. The rebound coefficient k is equal to -0.3.

This characteristic (a change in N or r simply accelerates or slows down the evolution) introduces a certain degree of flexibility: for example, if the number of bodies is too large (fragmentation after collision), we can decrease N to some convenient value, adjust r to conserve $N r^2$ and therefore continue to follow the

system economically. The decrease of N is simply obtained by elimination of a number of bodies chosen at random; this does not change the statistical properties of the system. The evolution and evolutionary time of some rather complicated system can in this way be made to correspond to those of a small system containing, say, a few hundred particles. Therefore, it is not the limitations of the numerical model which will determine the physical significance of the results.

This numerical simulation applies particularly well to astrophysical problems in which a given particle does not suffer too many collisions per revolution. The model is subject to the following considerations:

(1) If the mean free path λ of a given particle is small ($\lambda \ll R$), the effect of a collision is local, and the system resembles a continuous medium. It behaves as a fluid and is better described by local state variables rather than by particles and orbits. The system then presumably can be described by conventional hydrodynamical equations. In this case, there exist simpler and more powerful algorithms.

(2) The simulation is thus particularly useful for systems such that $\lambda \geqslant R$.

(3) Systems in which $\lambda \geqslant R$ are physically similar to stellar systems. They are in some sense "rarefied". A collision can take place anywhere along the orbit and the system is physically different from case (1): there is no longer a local equation of state, but only a more complicated "global equation".

(4) Our models all satisfy the condition $\lambda \geqslant R$.

(5) We have chosen N and r in such a way that a particle will on the average undergo one collision per revolution. There is no need to reduce the collision frequency any farther: if we do this, we merely increase the computing time without changing in any way the significance of the physical results of our model. The orbit of a particle does not change between two collisions and a collision can take place anywhere along the orbit, and so it does not matter how many times the particle goes round between two collisions.

Let w be the mean relative velocity of two particles. The average density of the particles is of the order of $N/((4/3)\,\pi R^3)$. A particle suffers a collision when its distance from another particle (measured centre to centre) is less than $2r$. Therefore, the number of collisions undergone by one particle per unit time is of the order of $3N(r/R)^2(w/R)$. Initially, the system is not flattened, w is roughly equal to the orbital velocity, and so w/R is roughly equal to the mean angular velocity at distance R. Thus a particle suffers roughly $3N(r/R)^2$ collisions per radian. Let us choose N, r and R in such a way that one particle has on the average one collision per revolution. For $N = 100$, this gives $r/R \simeq 0.023$. We note that the condition (5) is amply satisfied: the occupation coefficient $N(r/R)^3$ (i.e., the ratio of the volume of all particles to the total volume) is of the order of 10^{-3}. Finally, for this value of r/R, the distortion effect mentioned earlier is probably of negligible importance.

This demonstration is no longer valid when the system has flattened, but experience shows that the collision frequency decreases, and, a fortiori, the condition $\lambda \geqslant R$ is still satisfied.

III. The Initial Distribution and Choice of Units

The initial conditions of each particle are set up by selecting at random the six elements of the Keplerian orbit [21] in such a way that the trajectory is an ellipse lying between two spheres of radii R_1 and R_2 ; the inclination is chosen so as to lie between 0 and some upper limit i_{max} (see below). With this prescription, particles cannot approach too close to the central mass point nor escape before undergoing at least one collision. Let a be the semi-major axis, and let e be the eccentricity of an ellipse; our conditions are

$$\begin{aligned} a(1 - e) &\geqslant R_1 \\ a(1 + e) &\leqslant R_2 . \end{aligned} \tag{6}$$

They define a region (Z) in the (a, e) plane (Fig. 3). We thus select two random numbers e and a, uniformly distributed between 0 and 1, and R_1 and R_2, respectively, using one of the pseudo-random number generators recommended by Coveyou and MacPherson [22]. If the corresponding point falls inside (Z), the values of a and e are adopted; otherwise new random numbers are generated. It would be easy to choose initial conditions in different ways for specific applications.

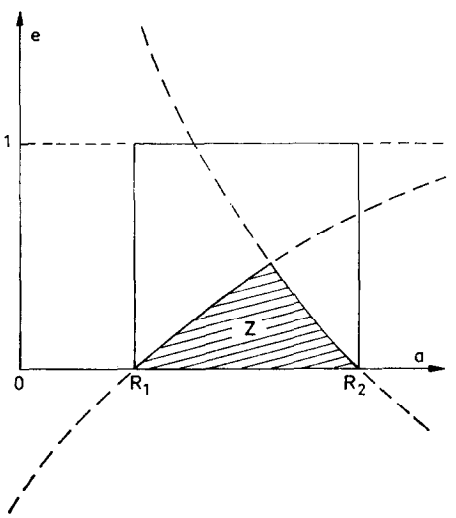

FIG. 3. The region (Z) in which the initial pairs (a, e) are taken; e: eccentricity; a: semi-major axis.

We start from a system relatively close to real physical cases, letting i be less than a given value i_{max}, which lies between 0 and π. In order that the orientations of the orbital planes be initially uniformly distributed in space, the initial orbital inclinations i are selected so that $\cos i$ is uniformly distributed in the range 1 to $\cos i_{max}$. Note that i_{max} is related to the initial angular momentum A: $\langle A \rangle$ is proportional to $(\cos(i_{max}/2))^2$ [5]. For a value of i_{max} close to π, this gives an almost spherical system with both direct and retrograde orbits; at the other extreme, small values of i_{max} correspond to flattened systems with all particles rotating in the same direction.

The initial longitude Ω of the ascending node, the argument ω of the pericentre, and the true anomaly ν are all selected at random from a uniform distribution between 0 and 2π. Now, in fact, the velocity of a particle varies along its orbit, and in principle it would have been better to consider the initial mean anomaly of a particle (which is proportional to the time) as being uniformly distributed, rather its true anomaly (i.e., its angular position). But experience shows that the results are not affected; furthermore, for small eccentricities, the true anomaly is roughly equal to the mean anomaly.

Of course, a new particle is not allowed to penetrate any other.

The unit of length is taken to be of the order of the dimension of the system in relation with the value of R_1 and R_2. Usually R_1 is put equal to 1. The central mass and the gravitational constant are put equal to unity.

This determines the unit of time: it is the time taken by one particle, on a circular orbit of radius the unit of length, to turn through an angle of one radian.

After the initial values of the six elements a, e, i, Ω, ω, and ν of the Keplerian orbit of each particle have been chosen, the position, velocity and acceleration vectors are calculated by classical formulae [21].

This entire initialisation procedure requires 2.5 sec. for 100 particles on an IBM 360-65.

IV. Calculation of Positions and Velocities at any Given Time

In order to identify collision pairs with the algorithm used here (see Section V), it is necessary to calculate the position, velocity and acceleration vectors at any given moment for all particles. These quantities in general cannot be expressed analytically as a function of time, and so we are obliged to solve Kepler's equation:

$$M = 2\pi(t - t_p)/P = E - e \sin E \tag{7}$$

where P is the period, t_p the time at which the particle is at the pericentre, E the eccentric anomaly and M the mean anomaly.

We use Newton–Raphson's iterative method [23] for calculating E, knowing t. Now, e lies in the range 0 to 1, and so the form of the relation between M and E is

such that a good starting point for the iteration is $E_0 = \pi$. In fact, we can save one step, because, in this case, E_1 is equal to $(M + \pi e)/(1 + e)$. We can therefore start immediately from E_1. The general iteration formula is:

$$E_{i+1} = E_i - ((E_i - e \sin E_i - M)/(1 - e \cos E_i)). \tag{8}$$

The iterations are continued until $| E_{i+1} - E_i| \leqslant |E_{i+2} - E_{i+1}|$, i.e., until no further improvement in accuracy is gained. The solution is E_{i+1}; with an IBM 360-65, this always corresponds to a quality of convergence $|E_{i+1} - E_i| \leqslant 2.10^{-6}$. Experience shows that the method is very fast—it converges after three to five iterations. The position, velocity and acceleration vectors are then calculated: this (together with the above iteration) requires 1.8 msec. per particle on an IBM 360-65.

Although the problem of solving Kepler's equation is classical, its solution in our case is not trivial because eccentricities can take any value between 0 and 1. Most methods are limited to moderate values of e, and it is for this reason that we have used Newton–Raphson's method. It works very well over the whole range of e from 0 to 1, and not only for e close to 0. Furthermore, thanks to the concavity of the curve $M = \phi(t)$, we can gain one step.

We can still use this iteration method even if the central body which is the source of the gravitational field is slightly oblate or inhomogeneous. In this case, the movement of each particle can still be described by an elliptical orbit with a slow drift of Ω, ω and M, given by classical equations [21]. This method applies equally well to the solar system or to Saturn's Ring. However, in the case of a galactic field or in the more complicated case of a self-gravitating system where the mutual attraction of masses is not neglected, the two-body equations cannot be used any more, and one must integrate the differential equations of motion for each particle.

V. The Algorithm for Identifying Collisions

The principal difficulty which appears in the calculations is to find whether two given particles will in fact collide or not. There is no explicit analytic relation giving the distance between two particles as a function of time, and so we have been led to use an approximate method.

In the field of molecular dynamics, Alder and Wainwright [7] have written an algorithm to calculate exactly the behaviour of several hundred interacting, classical particles. The algorithm first sets up a list of all possible two-particle collisions. It then finds which collision is the earliest, and the new velocity vectors of these particles are calculated. The list is then modified: all entries involving either of these two particles are deleted, and the collisions which now become possible are added. The algorithm then continues as above. This method applies particularly

well to molecular dynamics, because successive binary collisions are connected by straight-line segments.

The gravitational case is more complicated. In particular, as a consequence of the gravitational field, the particle trajectories are elliptical (standard model), or more complicated still (as would be the case for a self-gravitating system). The procedure used by Trulsen [13] is somewhat analogous to that of Alder and Wainwright: he also sets up a list of all collisions possible at a given moment. This is done by considering the elliptical tubes which are swept by each particle in space. The minimum distance between points on two elliptical orbits is calculated. If this distance is smaller than the sum of the particle radii, a collision is said to be possible, and the time at which this collision would occur is calculated.

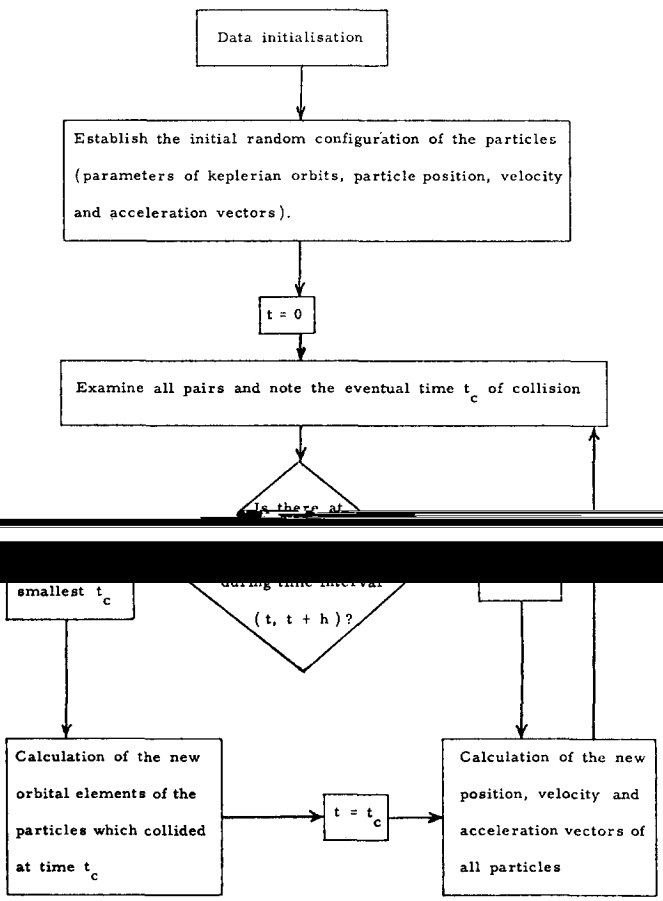

FIG. 4. General flow diagram.

We shall now describe a more general method, which also can be used conveniently even when the particle orbits are not ellipses. The general method is summarized in the flow diagram shown in Fig. 4.

We note that:

(1) We know at the time $t = t_0$ the values $\mathbf{r}_i$, $\dot{\mathbf{r}}_i$, and $\ddot{\mathbf{r}}_i$ of the position, velocity and acceleration vectors respectively of each particle (i).

(2) Consider a particle pair (i, j) during the time interval $(t_0, t_0 + h)$. The problem is to find if these two given particles collide during this time interval. We say that two particles collided during a given interval of time if the distance between the two centres became less than the sum of the particle radii.

(3) Define a function ϕ such that:

$$\phi(t) = (\mathbf{r}_i - \mathbf{r}_j)^2 - (2r)^2. \tag{9}$$

(4) A collision then corresponds to the first root of ϕ in the interval $(t_0, t_0 + h)$. This function ϕ can be expanded as a polynomial in terms of time: if we consider sufficiently small time intervals of length h, we need only keep terms up to second order.

(5) Consider the second-order expansion of ϕ with respect to time around the time t_1 in the interval $(t_0, t_0 + h)$:

$$\phi^*(t_1, t) = (\dot{\mathbf{r}}_{ij}^2 + \mathbf{r}_{ij} \cdot \ddot{\mathbf{r}}_{ij})(t - t_1)^2 + 2\mathbf{r}_{ij}\dot{\mathbf{r}}_{ij}(t - t_1) + \mathbf{r}_{ij}^2 - (2r)^2, \tag{10}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and values of $\mathbf{r}_{ij}$, $\dot{\mathbf{r}}_{ij}$, and $\ddot{\mathbf{r}}_{ij}$ are taken at the time $t = t_1$.

(6) The exact moment at which a collision occurs is found iteratively. The basic flow of this iterative search is shown in Fig. 5, and the procedure can be clarified by the following remarks:

(1) The iteration starts at $t_1 = t_0$.

(2) The calculation is considered to have converged when the difference between two successive iterations $|t_2 - t_1|$ is sufficiently small. However, we still check whether the result obtained does represent a real root of ϕ, and not simply a minimum. Furthermore, we require that the collision time found be included in the interval chosen.

(3) The "safety factors" m_1 and m_2 are included in order to allow for the crudeness of the second-order expansion.

(4) This calculation is done for each pair of particles, and we determine which particle pair would collide first during this particular interval. New orbital elements are then calculated for the two colliding particles, and we continue the algorithm from the current collision time. If during some time interval no collisions occur, the algorithm continues the search in the next time interval, and so on.

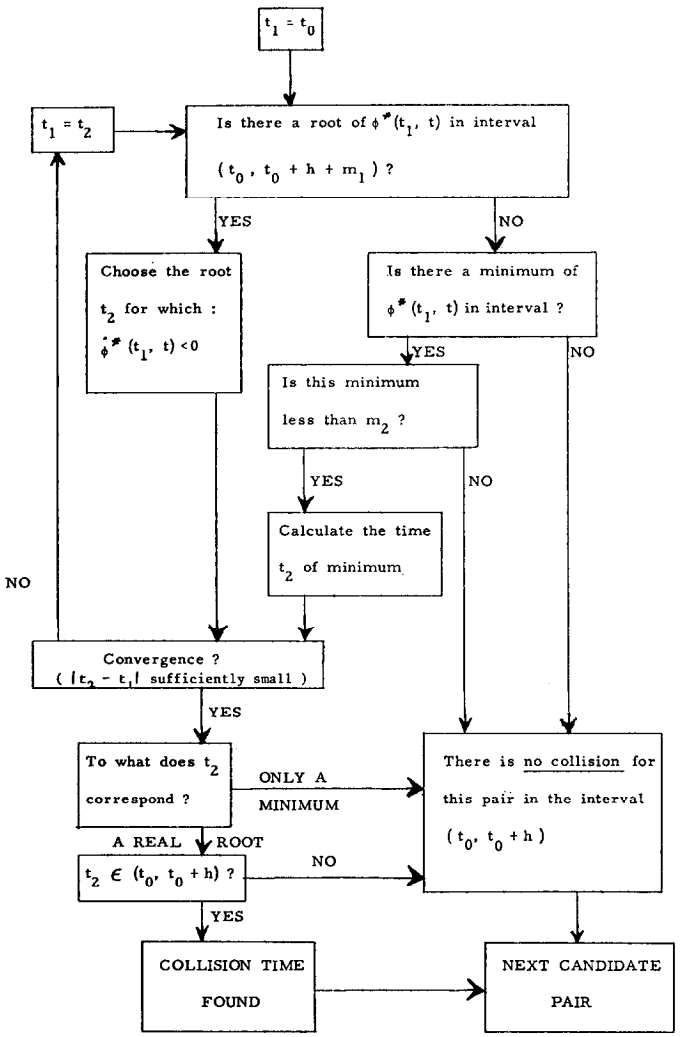

FIG. 5. Flow diagram showing how collisions are identified.

(5) The second-order approximation is used only for the search: when convergence has occured, t_1 is in principle the *exact* collision time.

(6) Choosing a large value of h saves the computing time, but it was found that the results start to deteriorate at h of the order of 1. m_1 and m_2 were empirically determined in a similar way by requiring reasonable accuracy and not inordinate computing time. $h = 0.4$, $m_1 = h/2$ and $m_2 = 12\ r^2$ are satisfactory for $N = 100$,

$r = 0.07$, $R_1 = 1$ and $R_2 = 3$. With the value of h used here, five or six iterations are usually sufficient to find the collision time of two particles. These calculations are very fast using the iterative solution of Kepler's equation (see Section IV). With 100 particles, the collision time between t and $t + h$ is found in about 0.56 sec on an IBM 360-65. Nevertheless, the collision search still takes 70 % of the overall computing time (see Section VII) and elimination procedures are therefore very important.

Two simple tests can indeed limit the number of collision candidates significantly:

(1) Consider the distance δ_1 of the pericentre of particle (i) and the distance δ_2 of the apocentre of particle (j) from the centre of the system. If $\delta_1 - \delta_2 > 2r$, no collision is possible between particles (i) and (j). We may therefore eliminate immediately particle pairs (i, j) such that:

$$a_i(1 - e_i) > a_j(1 + e_j) + 2r. \tag{11}$$

As long as two particles (i) and (j) do not collide, the inequality (11) is satisfied and there is no need to make this test at each step. It is sufficient to store the results. This test is particularly efficient and, indeed, its capacity for eliminating pairs improves during the course of the calculation. For instance, for $N = 100$, $r = 0.07$, $R_1 = 1$, and $R_2 = 3$ only one pair out of three satisfies equation (11) initially, but after one thousand collisions, one pair in two can be eliminated in this way.

(2) One can eliminate particle pairs for which the separation at the time t_0 is larger than some value δ which is determined by experience: these particles will not collide during a given time step h. This test, which must be done at the beginning of each step, usually eliminates one-third of the pairs surviving the first test.

The strategy described above does in principle waste a certain amount of computing time. It should be possible in principle to do better. Indeed one could avoid having to recalculate the same collision more than once by storing all possible collisions. This strategy would be efficient only if many collisions occur during the step h: this is in fact the situation for the short initial phase during which the system flattens. After the flattening, however, the collision rate decreases [3–5] and there is on the average less than one collision during a time interval h. Consequently, this improvement, which would complicate the algorithm, has no significant effect during that part of the calculation which takes longest, and so it is not really very useful.

Quentrec and Brot [24] observed that, in molecular dynamics problems, a considerable amount of the computing time and storage used in searching for the "neighbours" of a given particle could be saved by sorting particles into cells. In our case however, this method would not be useful because the number of particles involved is much smaller than in molecular dynamics.

A powerful check for the detection of programming errors [25] consists in verifying at the beginning of each step that the distance between any pair of particles has not become smaller than the sum of their radii: finding such an "interpenetration" indicates that a previous collision has been missed.

VI. The Collisions

The simulation program can be easily adapted to different types of particle collisions. In the standard model, the perpendicular component of the relative velocity of the colliding particles is multiplied by a coefficient k which lies between 0 and -1, whilst the grazing component is conserved. Thus, $k = -1$ corresponds to the elastic case. Particle spins have been neglected, and each collision is assumed to be instantaneous. This collision model is described in more detail elsewhere [5]. After each collision, we calculate the new elliptical elements of each particle.

In a forthcoming paper, we will study the case where the grazing component is not conserved. We hope in the future to introduce more realistic collision models, including in particular, fragmentation, coalescence, some kind of particle penetration; we shall also study the case where the rebound coefficient k depends on the relative velocity.

In the standard model, a special case can arise if the relative velocity of two colliding particles (i, j) is very small with respect to the orbital velocity and if the rebound coefficient k is small (for example, between 0 and -0.3). One finds that two such particles can continue very close together on almost parallel orbits—they drop into a kind of "permanent collision" and almost all of the computing time is spent just calculating and recalculating successive collision times between them. Unless a third particle knocks out one of these two (as in the French game of "pétanque" [26]) this state of affairs can continue for a long time and a considerable amount of computing time can be lost: the system, however, will not have evolved significantly. To avoid this circumstance, we make the rebound elastic ($k = -1$) if the relative velocity of two colliding particles is very small, say, less than one tenth of the orbital velocity. Intuitively, this would seem physically reasonable: it does not change appreciably the macroscopic evolution of the system and avoids numerical problems arising from the "permanent collision". This procedure is invoked about once every 100 collisions.

Even with inelastic collisions one of the particles involved can acquire a hyperbolic velocity. For a typical value of $k = -0.3$, this happens on the average once every 1000 collisions. To simplify the program we have assumed that particles on hyperbolic trajectories escape at once. We might in this way miss a collision during the time the escaping particle leaves the system on its hyperbolic orbit: however, this has practically no effect on the macroscopic evolution of the system.

If a particle is very close to the centre, its orbital period is very short and we need a very small step h. Also, the "permanent collision" phenomenon happens very [illegible] particle to cross the centre of mass. To avoid such problems, we assume that the central body which creates the gravitational field is a finite sphere of radius R_c which absorbs particles falling onto it. A particle's mass is assumed to be negligible with respect to the central mass. Thus, the absorption of a particle by the central body does not change the potential field. Experience shows that a good value for R_c is 0.2 (for $r = 0.07$, $R_1 = 1$, $R_2 = 3$). For $k = -0.3$, about one quarter of the particles has fallen onto the central body after 2000 collisions.

If the number N of particles becomes too small, statistical fluctuations become important (see Section II). It is possible to keep N constant. One way of doing this is by "creating" a new particle each time one escapes from the system or falls onto the central body. However, orbits of the "new particles" introduced in this way should have the statistical properties of the system at the time that they are introduced: we do this by selecting at random one of the existing particles, and placing the "new particle" somewhere at random along its orbit with a mean anomaly randomly chosen between 0 and 2π. Of course, the "new particle" is not allowed to penetrate any other.

The effect of one collision (including the calculation of new orbital elements) takes 61 msec on an IBM 360-65.

VII. Practical Considerations

(a) The evolution of all dynamical quantities of interest is surveyed by taking "snapshots" of the system from time to time.

After each collision, a number of interesting quantities are recorded: time, number of collisions per particle, mean eccentricity $\langle e \rangle$, mean inverse semi-major axis $\langle 1/a \rangle$, mean inclination $\langle i \rangle$, absolute value of the angular momentum, total energy lost through the collisions, energy lost by the bodies fallen onto the central mass, energy carried out by the escaped bodies, total energy, number of escaped bodies, number of bodies fallen onto the central mass, mean orbital period, mean size $\langle | x | \rangle$ of the system and dispersion $(\langle | x |^2 \rangle - \langle | x | \rangle^2)^{1/2}$, mean square velocity of all the particles $\langle | v |^2 \rangle$, and mean relative velocity of colliding particles.

After a given number n of collisions (usually 50), the mean value of these quantities over the n collisions is calculated and printed.

Every 300 collisions, a more detailed information is printed in addition to the quantities written above: number of steps, three components of the angular momentum, mean kinetic energy of the particles in the vertical, radial and tangential

directions, total potential energy, distributions of particles with respect to: distance from the horizontal plane, distance from the axis, distance from the centre of the system, and distributions of the orbital elements a, e, and i.

Every 1200 collisions, microscopic informations concerning individual particles are printed: position and velocity vectors, distance from the centre of the system, period, energy, orbital elements: a, e and i, and number of collisions suffered by the particle since the beginning.

(b) A check on the accuracy of the computation is that the total energy per unit mass (the sum of the actual energy of the particles, the energy lost by collision and the energy lost by particles removed from the system) should be conserved. Now, as a consequence of computer rounding errors, the orbital parameters calculated after each collision are slightly in error: these errors accumulate for successive collisions, and the total energy changes slowly. On the other hand, in the standard model, the calculation of the position and velocity of each particle in between collisions does not lead to such cumulative errors. This latter would not have been true had we integrated the differential equations of motion. After 5000 collisions, the relative error in the total energy is less than 10^{-5}: this allows us to follow the evolution of the system to its bitter end, the only limitation being the computing time available. The calculations were made in single precision with seven significant figures.

(c) The computing time could perhaps be reduced by using a Monte-Carlo method. Such a computation presupposes, however, that one knows the collision probability between particles (i) and (j) at a given time. This probability in principle can be correctly derived only from a detailed study (such as the one described here) of the dynamical evolution of the system.

(d) Linhart [27] noted the existence of "uncertainty epidemics" among interacting particles and gave a few examples for some simple dynamical systems. Our system exhibits the same kind of phenomenon: any initial small uncertainty (or any initial small perturbation) changes the future evolution of every particle. Indeed, if the position or the velocity of a particle (i) is changed very slightly, either the particle (i) will miss a future collision with the particle (j), or it will collide with a new particle (l): this changes completely the future of particles (i), (j), and (l) and very soon the orbit of every particle is completely different from what it would have been otherwise. One might ask if this does not destroy completely the validity of numerical experiments [28]. Fortunately, there are reasons to think that, although orbits become completely different, the macroscopic properties of the system are not affected [27]. We have verified numerically that, even though the individual behaviour of each particle is completely changed, the overall statistical properties of the system are not. However, as a consequence it is difficult to compare in detail results from different computers (which have different rounding error procedures).

(e) The collision search takes 69 % of the overall computing time. The effect of one collision including the calculation of new orbital elements takes 5 % and the rest is taken by calculating the new positions and velocities of all the particles at the beginning of each step. The computing time of initial conditions is negligible with respect to the rest of the calculation.

The program was debugged on the IBM 360-65 of the Institut National d'Astronomie et de Géophysique in the Observatory of Paris at Meudon and the results were obtained on the faster IBM 370-168 of the Centre National de la Recherche Scientifique in Orsay (C.I.R.C.E.). To follow 3000 collisions with 100 particles, we use 84 K octets of memory and 10 min computing time on the IBM 370-168.

VIII. Future Work

The present method is very general and should in principle allow a simulation of many astrophysical problems. The results for the three-dimensional and the two-dimensional cases are given elsewhere [5]. It would be interesting to take into account the following effects and models:

(a) The oblateness of the central body.

(b) The effect of a large body revolving around the system (for instance a satellite, in the case of Saturn's rings).

(c) A more realistic gravitational field (for example, for the galactic case).

(d) A self-gravitating system, using some kind of N-body integration, for example, Aarseth's method [29].

(e) Different collision models (grazing component not conserved, fragmentation, coalescence, etc., see Section VI).

(f) A distribution of particle sizes and masses.

These changes should not affect the basic structure of the program: in particular, the way in which collisions are found does not change.

A listing of the program may be obtained from the author.

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